

A NOTE ON THE MANIN-MUMFORD CONJECTURE

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ABSTRACT. We prove a variant of the Manin-Mumford conjecture for abelian schemes over a normal base scheme of characteristic zero. The proof is reduced to the Manin-Mumford conjecture over fields of characteristic zero, through a theorem of Grothendieck on the endomorphisms of abelian schemes. The theorem implies a case of the André-Oort conjecture for Kuga varieties, without resorting to the O-minimality approach nor the ergodic-Galois approach.

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INTRODUCTION

In this paper we discuss a variant of the Manin-Mumford conjecture for abelian schemes and its relation to the André-Oort conjecture for Kuga varieties.

Conjecture 0.1 (Manin-Mumford). *Let A be an abelian variety over \mathbb{C} , with $(a_i)_{i \in I}$ a family of torsion points. Then the Zariski closure of $\{a_i\}_{i \in I}$ is a finite union of torsion subvarieties, i.e. subvarieties of the form $A' + a'$ where $A' \subset A$ is an abelian subvariety and $+a'$ stands for the translation by some torsion point $a' \in A(\mathbb{C})$.*

Note that we may replace (a_n) by a sequence of torsion subvarieties, because a torsion subvariety is the Zariski closure of the set of torsion points in it.

The conjecture was first proved by M. Raynaud using p -adic methods, cf. [21], [22]. When A is defined over a number field, Faltings proved the more general Mordell-Lang conjecture which implies the Manin-Mumford conjecture, cf. [8], [9], as well as [10]. There have been many other proofs, like the ergodic-Galois approach in [20], the model-theoretic approach of E. Hrushovski [3], and the o-minimality approach by J. Pila and U. Zannier [17].

In [19], R. Pink has proposed a conjecture for mixed Shimura varieties as a combination of the André-Oort conjecture and the Mordell-Lang conjecture. It is further generalized into the Zilber-Pink conjecture. In this paper, we restrict our attention to a principal case of the conjecture of Pink which combines the André-Oort conjecture with the Manin-Mumford conjecture:

Conjecture 0.2 (André-Oort conjecture for Kuga varieties). *Let M be a Kuga variety, and let (M_i) be a family of special subvarieties in M . Then the Zariski closure of $\bigcup_i M_i$ is a finite union of special subvarieties in M .*

Here Kuga varieties M appear in the form of an abelian scheme $\pi : M \rightarrow S$ where S is a pure Shimura variety, typically corresponding to some moduli problem of abelian varieties, and M is the

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universal family of abelian varieties over S . Special subvarieties in M arise from diagrams of the following form

$$\begin{array}{ccccc} M' & \xrightarrow{\subset} & M_T & \xrightarrow{\subset} & M \\ & & \downarrow \pi|_T & & \downarrow \pi \\ & & T & \xrightarrow{\subset} & S \end{array}$$

with $T \subset S$ a moduli subspace (corresponding to abelian varieties with finer additional symmetries), $M_T \rightarrow T$ is the abelian T -scheme pulled-back from $M \rightarrow S$, and $M' \subset M_T$ is a special subscheme in the sense of ?? below, which is "roughly" an abelian subscheme translated by some torsion section.

Of course one may replace Kuga varieties by general mixed Shimura varieties. But the technique and results in this paper mainly focus on abelian schemes and Kuga varieties.

There have been remarkable progresses towards the André-Oort conjecture, cf. [24] for the ergodic-Galois approach and cf. [23] for a survey of the o-minimality approach of J. Pila. In the case of mixed Shimura varieties, [4] has proved the equidistribution of certain families of special subvarieties in Kuga varieties, and Z. Gao has proved the André-Oort conjecture for general mixed Shimura varieties whose pure part are Siegel modular varieties A_g , cf.[11]. The result of Gao is unconditional for $g \leq 6$, and relies on the GRH for CM fields when $g > 6$, as a generalization of previous results by J. Pila, J. Tsimerman, etc.

In this paper we prove the following statement:

Theorem 0.3 (main theorem). *Let $\pi : M \rightarrow S$ be a Kuga variety fibred over a pure Shimura variety, with (M_n) a sequence of special subvarieties such that $\pi(M_n) = S$ for all n . Then the Zariski closure of $\bigcup_n M_n$ is a finite union special subvarieties whose images under π are equal to S .*

It relies on a relative version of the Manin-Mumford conjecture for abelian schemes over a normal base scheme of characteristic zero. Although the arguments are elementary, even without the estimation of degrees, Galois orbits, etc., it does imply unconditionally a special case of the André-Oort conjecture for general Kuga varieties, which is not fully covered in [4] and [5]. We hope that it is useful as a footnote to the André-Oort conjecture.

The paper is organized as follows. In Section 1, we recall the basic notions of abelian schemes, special subschemes, monodromy representations, etc. In Section 2 we prove a relative Manin-Mumford conjecture for abelian schemes using a theorem of Grothendieck. In Section 3, we recall the basic notions of fibred Kuga varieties, and their special subvarieties. In Section 4, we use some results in Hodge theory to show that special subvarieties in Kuga variety that are faithfully flat over the base Shimura varieties are exactly the special subschemes when we view the Kuga variety as the total space of an abelian scheme, which finishes the proof.

1. SPECIAL SUBSCHEMES IN ABELIAN SCHEMES

We recall some basic notions of abelian schemes, details of which can be found in [15].

Definition 1.1 (abelian schemes and endomorphisms). Let S be a scheme.

(1) An abelian S -scheme is a proper smooth S -scheme $\pi : A \rightarrow S$ equipped with a group law. The group law is necessarily commutative, and it is written additively.

(2) Let $A \rightarrow S$ be an abelian S -scheme. We write $\text{End}_S(A)$ for the ring of endomorphisms of the abelian S -scheme A , i.e. morphisms of the S -scheme A respecting the group law. We write $\mathbf{End}_S(A)$ of the étale sheaf $U \mapsto \text{End}_U(A_U)$. Similarly, we have the ring of endomorphisms of $A \rightarrow S$ up to isogeny, namely $\text{End}_S^\circ(A) := \text{End}_S(A) \otimes_{\mathbb{Z}} \mathbb{Q}$, and the étale sheaf $\mathbf{End}_S^\circ(A)$. In practice we only need these sheaves on the finite étale sites.

(3) Let $A \rightarrow S$ be an abelian S -scheme. An abelian S -subscheme is just a smooth closed S -subgroup of $A \rightarrow S$.

Definition-Proposition 1.2 (torsions and Tate modules). Let $A \rightarrow S$ be an abelian S -scheme of relative dimension g . We assume for simplicity that S is connected, and we fix a geometric point x of S . Write $\pi_1(S) = \pi_1(S, x)$ for the étale fundamental group of S with base point x .

(1) For an integer $N \neq 0$, we have the endomorphism $[N] : A \rightarrow A$, sending a section a to the N -th multiple $a + \cdots + a$ (N -fold).

The endomorphism $[N] : A \rightarrow A$ is always flat, and its kernel $A[N] := \text{Ker}[N]$ is a finite flat group S -scheme.

When N is invertible over S , $A[N]$ is finite étale over S . In this case, the group $\pi_1(S, x)$ acts on the fiber $A[N]_x$ respecting the group law, hence it defines a continuous representation $\rho[N] : \pi_1(S, x) \rightarrow \mathbf{GL}_{\mathbb{Z}/N}(A[N]_x)$, which we call the monodromy representation of $\pi_1(S, x)$ on the N -torsion points. The kernel of $\rho[N]$ is a normal cofinite subgroup of $\pi_1(S, x)$ corresponding to a finite étale Galois covering S_N of S . S_N is universal in the sense that if $T \rightarrow S$ is a morphism of schemes such that in $A_T \rightarrow T$ we have $A_T[N] \cong (\mathbb{Z}/N)_T^{2g}$ is a constant étale sheaf over T , then T factors through $S_N \rightarrow S$ uniquely.

(2) For ℓ a rational prime, we have the integral ℓ -adic Tate module $\mathbb{T}_\ell A = \varprojlim_n A[\ell^n]$, and the rational ℓ -adic Tate module $\mathbb{T}_\ell^\circ A = \mathbb{T}_\ell A \otimes_{\mathbb{Z}_\ell S} \mathbb{Q}_{\ell S}$. When ℓ is invertible on S , the action of $\pi_1(S, x)$ gives a continuous ℓ -adic representation $\rho_\ell : \pi_1(S, x) \rightarrow \mathbf{GL}_{\mathbb{Z}_\ell}(\mathbb{T}_\ell A_x)$, which is called the ℓ -adic monodromy representation of $\pi_1(S, x)$ for $A \rightarrow S$. Note that when ℓ is invertible on S , $\mathbb{T}_\ell A_x$ is isomorphic to \mathbb{Z}_ℓ^{2g} as a topological abelian group.

Similarly, when S is of characteristic zero, we have the total Tate module $\mathbb{T}A = \varprojlim_N A[N]$, and the adelic Tate module $\mathbb{T}^\circ A = \mathbb{T}A \otimes_{\hat{\mathbb{Z}}_S} \hat{\mathbb{Q}}_S$. We also have the continuous monodromy representation $\rho : \pi_1(S, x) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_x)$, with $\mathbb{T}A_x \cong \hat{\mathbb{Z}}^{2g}$.

In particular, the kernel of $\rho : \pi_1(S, x) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_x)$ corresponds to a pro-finite étale covering $\hat{S} \rightarrow S$, such that for any integer $N \neq 0$, $\hat{A}[N]$ is a disjoint union of N^{2g} sections, where $\hat{A} \rightarrow \hat{S}$ is the base change of $A \rightarrow S$ along $\hat{S} \rightarrow S$.

In the rest of the paper, we assume that S is an integral scheme of characteristic zero.

To formulate our main results, we need the following variants of torsion points and torsion subvarieties:

Definition 1.3 (special sections and special subschemes). Let $A \rightarrow S$ be an abelian scheme.

(1) A special section is the image of some morphism of the form $S_N \hookrightarrow A_N \rightarrow A$, where $S_N \rightarrow A_N$ is a torsion section of $A_N \rightarrow S_N$ following the notations in 1.2(1), and $A_N \rightarrow A$ is the natural projection from the base change $A_N = A \times_S S_N$. Using finite étale descent, one verifies easily that special sections of $A \rightarrow S$ are S -subschemes that are finite étale over S such that after some finite étale base change it splits into a disjoint union of torsion sections: if $S' \subset A$ is a special section, then its preimage along some $A_N \rightarrow A$ is the orbit of a torsion section under $\pi_1(S, x)$.

(2) A special subscheme is the image of some morphism of the form $B_N \hookrightarrow A_N \rightarrow A$ for some $N \in \mathbb{N}_{>0}$, where $A_N = A \times_S S_N$ as in (1), and $B_N = A'_N + t_N$, where $A'_N \hookrightarrow A_N$ is an abelian S_N -subscheme, and t_N is an N -torsion section of $A_N \rightarrow S_N$. Since the image of A'_N in A is an abelian S -subscheme A' , we may think of the special subscheme as the $\pi_1(S, x)$ -orbit of the translation of A' by some torsion section.

When the monodromy representation is trivial, special sections are exactly torsion sections, and we have

Lemma 1.4 (generic fiber). *Let S be an integral scheme of characteristic zero, and let $A \rightarrow S$ be an abelian S -scheme of relative dimension g . Write η for the generic point of S with function field F , and $\bar{\eta}$ the geometric point given by the separable closure \bar{F} of F . If the monodromy representation $\pi_1(S, \eta) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_\eta)$ is trivial, then we have a bijection between torsion sections of $A \rightarrow S$ and torsion points in A_η , sending a torsion section to its generic fiber.*

Proof. Then the monodromy representation of A_η factors as $\text{Gal}(\bar{F}/F) \rightarrow \pi_1(S, \eta) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_\eta)$, hence it is also trivial, and all the torsion points in $A_\eta(\bar{F})$ are defined over F . For each integer $N > 0$, the triviality of the monodromy representation implies that $A[N] \cong (\mathbb{Z}/N)_S^{2g}$ is a constant finite étale group, with $A[N](S) \cong (\mathbb{Z}/N)^{2g}$. In particular, shrinking to the étale open $\{\eta\} \hookrightarrow S$ gives the identity $A[N](S) \rightarrow A_\eta[N](\eta)$, which is the desired bijection, N being an arbitrary integer. \square

We also have the following elementary fact:

Lemma 1.5. *Let S be an integral scheme of characteristic zero, and let $A \rightarrow S$ be an abelian S -scheme of dimension g . Let N_n be a sequence of positive integers tending to infinity as n grows. Then the union $\bigcup_n A[N_n]$ is dense in A for the Zariski topology.*

Proof. The structure map $A \rightarrow S$ being of finite presentation, we may assume that S is noetherian and geometrically integral.

If S is a field, then we may further assume that it is embedded in \mathbb{C} . Then $\bigcup_n A[N_n](\mathbb{C})$ is dense in $A(\mathbb{C})$ for the analytic topology, hence $\bigcup_n A[N_n]$ is dense in A for the Zariski topology.

For S geometrically integral with generic point η and function field F , it is clear that the abelian variety A_η is dense in A for the Zariski topology. Since $A[N_n]_\eta = A_\eta[N_n]$, we see that $\bigcup_n A[N_n]_\eta$ is dense in A , hence the density of $\bigcup_n A[N_n]$. \square

2. EXTENSION OVER A NORMAL BASE

In this section, we fix S a normal integral scheme of characteristic zero, and we fix $A \rightarrow S$ an abelian S -scheme of relative dimension g . Write η for the generic point of S , and $\bar{\eta}$ its algebraic closure. Write $\pi_1(S, \bar{\eta}) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_{\bar{\eta}})$ for the monodromy representation, whose kernel corresponds to a profinite Galois cover \hat{S} over S . Since $A \rightarrow S$ is of finite presentation, we may assume that S is locally noetherian.

Note that $\hat{S} \rightarrow S$ is also normal, the proof of which is reduced to the finite étale case, using the following

Lemma 2.1. *Let A be a normal integral ring, on which a finite group G acts by automorphisms. Then the subring A^G fixed by G is normal.*

Proof. Write F for the fraction field of A , and E the fraction field of A^G . Then for any $a \in E$ integral over A^G , its integral equation with coefficients in A^G is an integral equation over A , hence $a \in E \cap A = A^G$. \square

The reason we choose to work over an integral normal base of characteristic zero is the following (cf. [12] Theorem and Corollary 4.2):

Theorem 2.2 (A. Grothendieck). *Let S be a locally noetherian integral scheme over a field of characteristic zero, with A, B two abelian S -schemes, ℓ a fixed rational prime.*

(1) *Let $u_\ell : \mathbb{T}_\ell A \rightarrow \mathbb{T}_\ell B$ a homomorphism of integral ℓ -adic Tate modules. If for some point $s \in S$ the homomorphism $u_\ell(s)$ comes from a homomorphism of abelian $k(s)$ -schemes $u(s) : A(s) \rightarrow B(s)$, then u_ℓ comes from some homomorphism $u : A \rightarrow B$, i.e. it lies in the image of the natural homomorphism $\mathrm{Hom}_S(A, B) \rightarrow \mathrm{Hom}_{\mathbb{Z}_\ell}(\mathbb{T}_\ell A, \mathbb{T}_\ell B)$.*

(2) *Assume moreover that S is normal, with U an open subscheme of S , and X an abelian U -scheme. Then X extends to an abelian S -scheme $\mathcal{X} \rightarrow S$ if and only if $\mathbb{T}_\ell X$ is unramified over S , in the sense that for any $n \in \mathbb{N}$, $X[\ell^n]$ extends to an étale cover of S .*

Proposition 2.3 (constant subsheaf). *Let $A \rightarrow S$ be an abelian S -scheme, with S normal integral, such that the monodromy representation $\pi_1(S, \bar{\eta}) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_{\bar{\eta}})$ is trivial, i.e. $S = \hat{S}$. Then the sheaf $\mathrm{End}_S(A)$ is a constant subsheaf of $\mathrm{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}A)$.*

Proof. $\mathrm{End}_S(A)$ is a subsheaf of $\mathrm{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}A)$, because for any étale morphism $U \rightarrow S$, $\mathrm{End}_U(A_U)$ is naturally a subset of $\mathrm{End}_{\hat{\mathbb{Z}}_U}(\mathbb{T}A_U)$: if a morphism $f : A_U \rightarrow A_U$ sends each N -torsion subgroup to zero, then it sends the closure of $\bigcup_N A_U[N]$ to zero, namely it is zero as an endomorphism of A_U over U .

For the constancy, we first show that any geometric point x over η gives an isomorphism $\tau : \mathrm{End}_S(A) \rightarrow \mathrm{End}_x(A_x)$ by restriction. The injectivity is proved as above, and for the surjectivity, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{End}_S(A) & \longrightarrow & \mathrm{End}_x(A_x) \\ \downarrow & & \downarrow \\ \mathrm{End}_{\hat{\mathbb{Z}}_S}(\mathbb{T}A) & \longrightarrow & \mathrm{End}_{\hat{\mathbb{Z}}}(\mathbb{T}A_x) \end{array}$$

where the horizontal map on the bottom is bijective due to the triviality of the monodromy representation. The two vertical maps are inclusions, hence the horizontal map τ on the top row is surjective, using 2.2 (1).

Now for any étale morphism $U \rightarrow S$ with a geometric point x in U , the monodromy representation $\pi_1(U, x) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}(A_U)_x)$ is trivial, hence $\mathrm{End}_U(A_U) \rightarrow \mathrm{End}_x(A_x)$ is bijective. Hence $\mathrm{End}_S(A) \rightarrow \mathrm{End}_U(A_U)$ is an isomorphism for all U , which proves the constancy. \square

For an abelian variety we can realize its abelian subvariety as the neutral component of the kernel of some endomorphism, based on the following:

Theorem 2.4 (splitting theorem, cf. [2] 3.19, 3.20). *Let k be a field, and let X be an abelian variety over k . Then for any abelian subvariety $Y \subset X$, there exists an abelian subvariety $Z \subset X$ such that the product map $Y \times Z \rightarrow X$ is an isogeny.*

In fact let $Y \subset X$ be an abelian subvariety, with N the degree of the isogeny $Y \times Z \rightarrow X$ given by the theorem. The multiplication $[N] : Y \times Z \rightarrow Y \times Z$ factors through some isogeny $(p_Y, p_Z) : X \rightarrow Y \times Z$, and the composition

$$X \xrightarrow{(p_Y, p_Z)} Y \times Z \xrightarrow{i_Y, i_Z} X \times X \xrightarrow{m_X} X$$

is an isogeny, with i_Y and i_Z inclusions of abelian subvarieties. In particular, the composition $\psi := m_X \circ (0 \times i_Z) \circ p_Z \in \mathrm{End}_k(X)$ is an endomorphism, whose kernel contains Y as the neutral component.

Corollary 2.5. *Let $A \rightarrow S$ etc. be as in the beginning of this section, with S normal integral. If the monodromy representation is trivial, then every abelian subvariety A' in the generic fiber A_η extends to an abelian S -subscheme B of $A \rightarrow S$ with $B_\eta = A'$.*

Proof. Let $A' \subset A_\eta$ be an abelian subvariety. Then by 2.4 we can find some endomorphism $\phi : A_\eta \rightarrow A_\eta$ such that A' is equal to the neutral component of the closed subgroup variety $\mathrm{Ker}\phi$. The constancy of $\mathrm{End}_S(A)$ shows that ϕ extends to a unique endomorphism Φ of $A \rightarrow S$. The kernel $\mathrm{Ker}\Phi$ is a closed S -subgroup of $A \rightarrow S$, and it is smooth over S because it is the pull-back of the neutral section $S \hookrightarrow A$ along Φ . Therefore the neutral component of $\mathrm{Ker}\Phi$, denoted as B , is a closed S -subscheme of A and is an abelian S -subscheme under the group law of $A \rightarrow S$. Taking generic fiber we see that B_η is a connected subgroup variety of $\mathrm{Ker}\Phi_\eta = \mathrm{Ker}\phi$, namely it is equal to A' . \square

We also have the following

Corollary 2.6 (descent to finite level). *Let $A \rightarrow S$ be an abelian S -scheme, with S normal integral of characteristic zero. Let $\hat{S} \rightarrow S$ be the pro-finite étale Galois covering corresponding to the kernel of the monodromy representation, and let $A' \rightarrow \hat{S}$ be an abelian \hat{S} -subscheme of $\hat{A} := A \times_S \hat{S}$. Then A' descends to some finite étale cover $T \rightarrow S$, i.e. there exists a finite étale cover $T \rightarrow S$ such that $\hat{S} \rightarrow S$ factors as $\hat{S} \rightarrow T \rightarrow S$ and that $A' = B_{\hat{S}}$ where B is an abelian T -subscheme of the base change $A_T \rightarrow T$.*

Proof. This is the standard reduction of projective limits: $A' \subset \hat{A}$ is the neutral component of $\mathrm{Ker}\phi$ for some endomorphism $\phi : \hat{A} \rightarrow \hat{A}$. Since the projective limit $\hat{S} = \varprojlim S_N$ is taken over the filtrant system (S_N) with S_N corresponding to the kernel of $\pi_1(S, \bar{\eta}) \rightarrow \mathbf{GL}_{\mathbb{Z}/N}(A[N]_{\bar{\eta}})$, there exists some integer $N > 0$ such that $\phi : \hat{A} \rightarrow \hat{A}$ is pulled-back from some endomorphism $\Phi : A_N \rightarrow A_N$ with $A_N = A \times_S S_N$, and that $\mathrm{Ker}\Phi$ has neutral component B such that B is an abelian S_N -subscheme with $B_{\hat{S}} = A'$. One may thus take $T = S_N$. \square

We proceed to prove the Manin-Mumford conjecture in the relative setting using special subschemes.

Proposition 2.7 (relative Manin-Mumford). *Let $A \rightarrow S$ be an abelian S -scheme, with S a normal integral scheme of characteristic zero. Let A_n be a sequence of special subschemes of $A \rightarrow S$. Then the Zariski closure of $\bigcup_n A_n$ can be represented as a finite union of special subschemes.*

Proof. Since $A \rightarrow S$ is of finite presentation, we may assume for simplicity that S is geometrically integral of generic point η , with $\bar{\eta}$ the geometric point in S corresponding to the separable closure of η .

(1) We first consider the case when the monodromy representation is trivial, i.e. $\hat{S} = S$. In this case, we have proved that taking base change from S to η gives

- a bijection between torsion sections of $A \rightarrow S$ and torsion points of A_η ;
- a bijection between abelian S -subschemes of $A \rightarrow S$ and abelian subvarieties of A_η .

And taking Zariski closure gives inverses to these bijections because η is dense in S .

Special subschemes in A are of the form $a+B$ with a a torsion section and B an abelian S -subscheme. Let A_n be a sequence of special subschemes of $A \rightarrow S$. Then $A_{n,\eta}$ is a torsion subvariety of A_η , with $A_{n,\eta}$ dense in A_n for the Zariski closure topology. The closure of $\bigcup_n A_{n,\eta}$ in A_η is a finite union of torsion subvarieties, whose closure is a finite union of special subscheme in A . The Manin-Mumford conjecture is thus immediate in this case.

(2) In general, a special subscheme of A is the image $\pi'(A')$ where

- $\pi' : A_N \rightarrow A$ is the projection for some base change $A_N \rightarrow S_N$, S_N corresponding to the kernel of $\pi_1(S, \bar{\eta}) \rightarrow \mathbf{GL}_{\mathbb{Z}/N}(A[N]_{\bar{\eta}})$;
- $A' = a' + B'$ for some torsion section a' of $A_N \rightarrow S_N$ and some abelian S_N -subscheme B' of A_N .

Hence a special subscheme is of the form $\pi(A')$, where $\pi : \hat{A} \rightarrow A$ is the projection for the base change $\hat{A} \rightarrow \hat{S}$, with \hat{S} corresponding to the kernel of $\pi_1(S, \bar{\eta}) \rightarrow \mathbf{GL}_{\hat{\mathbb{Z}}}(\mathbb{T}A_{\bar{\eta}})$, and $A' = a' + B'$ for some torsion section a' of $\hat{A} \rightarrow \hat{S}$ and B' some abelian \hat{S} -subscheme.

The projection $\pi : \hat{A} \rightarrow A$ is a pro-finite cover, and in particular it is universally closed. Let A_n be a special subscheme in A of the form $\pi(B_n)$ with B_n a special subscheme of $\hat{A} \rightarrow \hat{S}$. Then the Zariski closure of $\bigcup_n A_n$ contains $\pi(B)$, with B the Zariski closure of $\bigcup_n B_n$ in \hat{A} . By (1) we know that B is a finite union of special subschemes in \hat{A} , hence $\pi(B)$ is a finite union of special subschemes in A , hence it is equal to the Zariski closure of $\bigcup_n A_n$. \square

3. PRELIMINARIES ON KUGA VARIETIES

We recall briefly the definitions of Kuga data, Kuga varieties, and their special subvarieties, cf. [4] Section 2.

Definition 3.1 (Kuga data). A Kuga datum is a pair (\mathbf{P}, Y) given by some $(\mathbf{G}, X; \mathbf{V})$ as follows

- (\mathbf{G}, X) is a pure Shimura datum in the sense of [7]; in particular, X is a $\mathbf{G}(\mathbb{R})$ -conjugacy class of homomorphisms $x : \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}}$ subject to some algebraic constraints;
- $\rho : \mathbf{G} \rightarrow \mathbf{GL}_{\mathbf{V}}$ is an algebraic representation on a finite-dimensional \mathbb{Q} -vector space such that for any $x \in X$ the composition $\rho \circ x : \mathbb{S} \rightarrow \mathbf{GL}_{\mathbf{V}, \mathbb{R}}$ is a Hodge structure of type $\{(-1, 0), (0, -1)\}$.

We put $\mathbf{P} = \mathbf{V} \rtimes \mathbf{G}$ and $Y = \mathbf{V}(\mathbb{R}) \times X$, with Y viewed as a $\mathbf{P}(\mathbb{R})$ -conjugacy class of homomorphisms $y : \mathbb{S} \rightarrow \mathbf{P}_{\mathbb{R}}$ subject to some algebraic constraints. In the language of [5], $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ is fibred over (\mathbf{G}, X) .

When $\mathbf{V} = 0$, we get (pure) Shimura data.

For simplicity, we also require that the Kuga data are irreducible in the sense of [18] 2.13, which means that for any \mathbb{Q} -subgroup $\mathbf{H} \subsetneq \mathbf{G}$ there is some $x \in X$ such that $x(\mathbb{S}) \not\subseteq \mathbf{H}_{\mathbb{R}}$.

Definition 3.2 (morphisms and subdata). A morphism between Kuga data is of the form $(f, f_*) : (\mathbf{P}, Y) \rightarrow (\mathbf{P}', Y')$ with $f : \mathbf{P} \rightarrow \mathbf{P}'$ a homomorphism of \mathbb{Q} -groups, and $f_* : Y \rightarrow Y'$ is the push-forward sending $y : \mathbb{S} \rightarrow \mathbf{P}_{\mathbb{R}}$ to $f \circ y : \mathbb{S} \rightarrow \mathbf{P}'_{\mathbb{R}}$.

A subdatum of (\mathbf{P}, Y) is a morphism of Kuga data $(f, f_*) : (\mathbf{P}_1, Y_1) \rightarrow (\mathbf{P}, Y)$ such that both f and f_* are inclusions of subsets.

Let $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ be a Kuga datum. The natural map $(\mathbf{P}, Y) \rightarrow (\mathbf{G}, X)$ is a morphism of Kuga data, which we call the natural projection of (\mathbf{P}, Y) onto its pure base: \mathbf{G} is the maximal reductive quotient of \mathbf{P} . The Levi decomposition $\mathbf{P} = \mathbf{V} \rtimes \mathbf{G}$ also extends to an inclusion of subdatum $(\mathbf{G}, X) \hookrightarrow (\mathbf{P}, Y)$ which we call the pure section corresponding to $\mathbf{P} = \mathbf{V} \rtimes \mathbf{G}$.

Note that for a Kuga datum $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$, Y is a complex manifold with a transitive action of $\mathbf{P}(\mathbb{R})$, and the natural projection $Y \rightarrow X$ is a holomorphic vector bundle, equivariant with respect to

$\mathbf{P}(\mathbb{R}) \rightarrow \mathbf{G}(\mathbb{R})$. The fiber $\pi^{-1}x$ is the real vector space $\mathbf{V}(\mathbb{R})$ with the complex structure defined by $x : \mathbb{S} \rightarrow \mathbf{G}_{\mathbb{R}} \rightarrow \mathbf{GL}_{\mathbf{V}, \mathbb{R}}$.

Definition 3.3 (connected Kuga varieties). We write $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ for the ring of finite adeles.

(1) The (complex) Kuga variety defined by the Kuga datum (\mathbf{P}, Y) at level K for some compact open subgroup $K \subset \mathbf{P}(\hat{\mathbb{Q}})$ is a double quotient of the form

$$M_K(\mathbf{P}, Y)(\mathbb{C}) = \mathbf{P}(\mathbb{Q}) \backslash (Y \times \mathbf{P}(\hat{\mathbb{Q}})/K)$$

with $\mathbf{P}(\mathbb{Q})$ acts on $Y \times \mathbf{P}(\hat{\mathbb{Q}})/K$ through the diagonal. Take $\mathbf{P}(\mathbb{Q})_+$ the stablizer in $\mathbf{P}(\mathbb{Q})$ of some connected component $Y^+ \subset Y$, we have

$$M_K(\mathbf{P}, Y)(\mathbb{C}) = \coprod_a \Gamma_K(a) \backslash Y^+$$

with $\Gamma_K(a) = \mathbf{P}(\mathbb{Q})_+ \cap aKa^{-1}$, a running through a set of representatives of the finite double quotient $\mathbf{P}(\mathbb{Q})_+ \backslash \mathbf{P}(\hat{\mathbb{Q}})/K$.

The general theory of mixed Shimura varieties in [18] shows that the set $M_K(\mathbf{P}, Y)(\mathbb{C})$ defined above are quasi-projective normal varieties over \mathbb{C} , and they admits canonical models over certain number fields. In this paper we only treat them as normal algebraic varieties over \mathbb{C} .

The map $\wp_{\mathbf{P}} : Y \times \mathbf{P}(\hat{\mathbb{Q}})/K \rightarrow M_K(\mathbf{P}, Y)(\mathbb{C})$, $(y, aK) \mapsto [y, aK]$ is called the (complex) uniformization.

(2) A connected Kuga datum is of the form $(\mathbf{P}, Y; Y^+)$ with $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ a Kuga datum and $Y^+ \subset Y$ a connected component of Y . Note that Y^+ is homogeneous under $\mathbf{P}(\mathbb{R})^+$. We also have $(\mathbf{P}, Y; Y^+) = \mathbf{V} \rtimes (\mathbf{G}, X; X^+)$ in the sense of 3.1, with X^+ the image of Y^+ in X which is a connected component of X .

Connected Kuga varieties are quasi-projective algebraic varieties over \mathbb{C} of the form $\Gamma \backslash Y^+$ with $\Gamma \subset \mathbf{P}(\mathbb{Q})_+$ some congruence subgroup. They also admit canonical models over some number fields.

We write \wp_{Γ} for the uniformization map $Y^+ \mapsto \Gamma \backslash Y^+$, $y \mapsto \Gamma y$.

(3) In particular, when we write $(\mathbf{P}, Y; Y^+) = \mathbf{V} \rtimes (\mathbf{G}, X; X^+)$ and take a congruence subgroup of the form $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$, with $\Gamma_{\mathbf{V}} \subset \mathbf{V}(\mathbb{Q})$ and $\Gamma_{\mathbf{G}} \subset \mathbf{G}(\mathbb{Q})_+$ congruence subgroups such that $\Gamma_{\mathbf{V}}$ is stabilized by $\Gamma_{\mathbf{G}}$, then we have the natural projection $\pi : M = \Gamma \backslash Y^+ \rightarrow S = \Gamma_{\mathbf{G}} \backslash X^+$, which is an abelian S -scheme with neutral section $S \hookrightarrow M$ given by $(\mathbf{G}, X; X^+) \hookrightarrow (\mathbf{P}, Y; Y^+)$.

Assumption 3.4. Unless otherwise mentioned, we will always assume that $\Gamma_{\mathbf{G}}$ is a torsion-free congruence subgroup of $\mathbf{G}(\mathbb{Q})_+$. In this case S is smooth, and the natural map $\Gamma'_{\mathbf{G}} \backslash X^+ \rightarrow \Gamma_{\mathbf{G}} \backslash X^+$ is finite étale for any congruence subgroup $\Gamma'_{\mathbf{G}} \subset \Gamma_{\mathbf{G}}$. Since S is also normal by [1], we see that the étale fundamental group of S is equal to the pro-finite completion of Γ , the image of $\Gamma_{\mathbf{G}}$ inside $\text{Aut}(X^+) \cong \mathbf{G}^{\text{ad}}(\mathbb{R})^+$, which only differs from $\Gamma_{\mathbf{G}}$ by a central subgroup.

Remark 3.5 (group law). Let $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ be a fibred Kuga datum. We write the group law on $\mathbf{P} = \mathbf{V} \rtimes \mathbf{G}$ as

$$(v, g) \cdot (v', g') = (v + g(v'), gg')$$

for local sections $v, v' \in \mathbf{V}$, $g, g' \in \mathbf{G}$, with $g(v') = gv'g^{-1} = \rho(g)(v')$ by the representation $\rho : \mathbf{G} \rightarrow \mathbf{GL}_{\mathbf{V}}$. In particular, for $u \in \mathbf{V}$, we have $u(v, g)u^{-1} = (u, 1)(v, g)(-u, 1) = (v + u - g(u), g)$.

Write $\pi : (\mathbf{P}, Y) \rightarrow (\mathbf{G}, X)$ for the natural projection, then the fibred product $(\mathbf{P}, Y) \times_{(\mathbf{G}, X)} (\mathbf{P}, Y)$ exists as a fibred Kuga datum, which is simply $(\mathbf{V} \oplus \mathbf{V}) \rtimes (\mathbf{G}, X)$. The sum $\mathbf{V} \oplus \mathbf{V} \rightarrow \mathbf{V}$ defines a group law $(\mathbf{P}, Y) \times_{(\mathbf{G}, X)} (\mathbf{P}, Y) \rightarrow (\mathbf{P}, Y)$ with $(\mathbf{G}, X) \rightarrow (\mathbf{P}, Y)$ as the neutral section. On $\mathbf{P} = \mathbf{V} \rtimes \mathbf{G}$ it writes as $(v, g) + (v', g) = (v + v', g)$ and on Y it writes as $(v, x) + (v', x) = (v + v', x)$. Fix a connected component $X^+ \subset X$, its pre-image $Y^+ = \pi^{-1}X^+ \subset Y$, and congruence subgroups $\Gamma_{\mathbf{G}} \subset \mathbf{G}(\mathbb{Q})_+$, $\Gamma_{\mathbf{V}} \subset \mathbf{V}(\mathbb{Q})$ (stabilized by $\Gamma_{\mathbf{G}}$) and $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$, we see that $M = \Gamma \backslash Y^+ \rightarrow S = \Gamma_{\mathbf{G}} \backslash X^+$ is a bundle of compact Lie group over S : $(\bar{v}, \bar{x}) + (\bar{v}', \bar{x}) = (\overline{v + v'}, \bar{x})$ for $(v, x), (v', x) \in \pi^{-1}x, x \in X^+$.

The fibers are compact complex tori, and $M \rightarrow S$ is an abelian S -scheme as the variation of Hodge structures given by the monodromy representation $\pi_1(S) \rightarrow \mathbf{GL}_{\Gamma_{\mathbf{V}}}$ is polarized, due to the universal property of (\mathbf{G}, X) mentioned later in 4.2; see also [7], [18] and [19].

Definition 3.6 (special subvarieties and Hecke translates). For $M = \Gamma \backslash Y^+$ a connected Kuga variety defined by $(\mathbf{P}, Y; Y^+)$ as above, a special subvariety in M is of the form $\wp_\Gamma(Y'^+)$ given by some subdatum $(\mathbf{P}', Y'; Y'^+) \subset (\mathbf{P}, Y; Y^+)$. Note that we require Y'^+ to be a connected component of Y' contained in Y^+ .

Take $q \in \mathbf{P}(\mathbb{Q})_+$, $q\Gamma q^{-1}$ remains a congruence subgroup of $\mathbf{P}(\mathbb{Q})_+$, and we have an isomorphism $\tau_q : M = \Gamma \backslash Y^+ \rightarrow q\Gamma q^{-1} \backslash Y^+$, $\Gamma \cdot y \mapsto q\Gamma q^{-1} \cdot qy$, called the Hecke translation by q . Note that when $q \in \mathbf{V}(\mathbb{Q})$, $(\mathbf{P}, Y; Y^+) = \mathbf{V} \rtimes (q\mathbf{G}q^{-1}, qX; qX^+)$, and τ_q sends the pure section of $M \rightarrow S$ to the pure section of $M' = q\Gamma q^{-1} \backslash Y^+ \rightarrow S' = q\Gamma_{\mathbf{G}} q^{-1} \backslash qX^+$ given by $(q\mathbf{G}q^{-1}, qX; qX^+) \hookrightarrow (\mathbf{P}, Y; Y^+)$.

Of course we can also talk about more general Hecke translation given by $q \in \mathbf{P}(\hat{\mathbb{Q}})$, cf. [4].

The following proposition describes subdata and special subvarieties in an explicit way as we have seen in Introduction.

Proposition 3.7 (description of subdata and special subvarieties). (1) Let $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ be a Kuga datum fibred over a pure Shimura datum (\mathbf{G}, X) . Then a Kuga subdatum $(\mathbf{P}', Y') \subset (\mathbf{P}, Y)$ is of the form $(\mathbf{P}', Y') = \mathbf{V}' \rtimes (v\mathbf{G}'v^{-1}, vX')$ where (\mathbf{G}', X') is a pure Shimura subdatum of (\mathbf{G}, X) , \mathbf{V}' is a subrepresentation of \mathbf{G}' in \mathbf{V} , and $v \in \mathbf{V}(\mathbb{Q})$ conjugate \mathbf{G}' into a Levi \mathbb{Q} -subgroup $v\mathbf{G}'v^{-1}$ of \mathbf{P}' . For a fixed (\mathbf{P}', Y') , v is unique up to translation by $\mathbf{V}(\mathbb{Q})$.

(2) Let $M = \Gamma \backslash Y^+$ be a connected Kuga variety defined by $(\mathbf{P}, Y; Y^+) = \mathbf{V} \rtimes (\mathbf{G}, X; X^+)$ with $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$. The natural projection $\pi : M \rightarrow S = \Gamma_{\mathbf{G}} \backslash X^+$ defines an abelian S -scheme, and the special subvariety M' defined by $(\mathbf{P}', Y'; Y'^+) = \mathbf{V}' \rtimes (v\mathbf{G}'v^{-1}, vX'; vX'^+)$ fits into the diagram

$$\begin{array}{ccccc} M' & \xrightarrow{\subset} & M_{S'} & \xrightarrow{\subset} & M \\ & & \downarrow \pi & & \downarrow \pi \\ & & S' & \xrightarrow{\subset} & S \end{array}$$

where $S' = \wp_{\Gamma_{\mathbf{G}}}(X'^+)$ is the pure special subvariety in S defined by $(\mathbf{G}', X'; X'^+)$, $M_{S'}$ is the pull-back of $M \rightarrow S$ along $S' \rightarrow S$, equal to the special subvariety defined by $\mathbf{V} \rtimes (\mathbf{G}', X'; X'^+)$. M' is a torsion subscheme of the abelian S' -scheme $M_{S'} \rightarrow S'$ in the sense of ???: the subdatum $\mathbf{V}' \rtimes (\mathbf{G}', X'; X'^+)$ defines an abelian S' -subscheme $A'_{S'}$, and $(v\mathbf{G}'v^{-1}, vX'; vX'^+)$ defines a special section of $M_{S'} \rightarrow S'$, the "translation" by which gives the torsion subscheme M' .

Proof. The part (1) is from [4] 2.6 and 2.10. We only outline how (2) is interpreted via the special subschemes. We may thus assume that $S' = S$.

Write $\Gamma_{\mathbf{G}}(v) = \{g \in \Gamma_{\mathbf{G}} : v - g(v) \in \Gamma_{\mathbf{V}}\}$. Then $v\Gamma_{\mathbf{G}}(v)v = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}} \cap v\Gamma_{\mathbf{G}}v^{-1}$. Base change to $f : T = \Gamma_{\mathbf{G}}(v) \backslash X^+ \rightarrow S$, we get the abelian T -scheme $M_T = (\Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}(v)) \backslash Y^+ \rightarrow T$. Aside from the neutral section $T \hookrightarrow M_T$ given by $(\mathbf{G}, X; X^+) \subset (\mathbf{P}, Y; Y^+)$ and $\Gamma_{\mathbf{G}}(v) \subset \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}(v)$, we also have the pure special subvariety $T(v) = \wp_\Gamma(vX^+)$ corresponding to $(v\mathbf{G}v^{-1}, vX; vX^+)$. Since we have shrunk to $\Gamma_{\mathbf{G}}(v)$, the equality $v(\Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}(v))v^{-1} = \Gamma_{\mathbf{V}} \rtimes v\Gamma_{\mathbf{G}}(v)v^{-1}$ implies that $T(v) = v\Gamma_{\mathbf{G}}(v)v^{-1} \backslash vX^+ \cong \Gamma_{\mathbf{G}}(v) \backslash X^+$, and thus $T(v)$ is a torsion section, whose torsion order is the minimal integer $N > 0$ such that $N \cdot v \in \Gamma_{\mathbf{V}}$. The subdatum $\mathbf{V}' \rtimes (\mathbf{G}, X; X^+)$ defines an abelian T -subscheme of M_T , whose translation by $T(v)$ is a torsion subscheme of M_T . Its image under $M_T \rightarrow M$ is a special subscheme of M , which is exactly the special subvariety $M' = \wp_\Gamma(Y'^+)$. \square

4. SPECIAL SUBSCHEMES IN KUGA VARIETIES

In this section we show that special subschemes in a fibred Kuga variety $M \rightarrow S$ are special subvarieties that are faithfully flat over S . The proof makes use of some facts from the theory of variation of Hodge structures, details of which can be found in [7], [13], [14] etc. We adopt standard abbreviations such as "HS" for Hodge structures, "PVHS" for polarized variation of Hodge structures, etc.

Theorem 4.1 (abelian schemes vs. variation of Hodge structures, [6] 4.4.3(a)). *Let S be a smooth scheme over \mathbb{C} of finite type. Then we have the equivalence between the following two categories:*

- (1) the category of abelian S -schemes (with morphisms respecting the group laws);
- (2) the category of polarizable variation of integral Hodge structures (\mathbb{Z} -PVHS) of type $\{(-1, 0), (0, -1)\}$.

The equivalence sends an abelian S -scheme $f : A \rightarrow S$ to the \mathbb{Z} -PVHS $\mathcal{H} = \mathcal{H}(A/S)$ whose underlying local system of \mathbb{Z} -modules is dual to $R^1 f_* \mathbb{Z}_A$, with $\mathcal{H}_s = H_1(A_s, \mathbb{Z})$ as the fiber at s . The exponential map realize A as the the quotient sheaf

$$0 \rightarrow \mathcal{H} \rightarrow \text{Lie}_S A \rightarrow A \rightarrow 0$$

where $\text{Lie}_S A$ is the sheaf of "vertical tangents" of $A \rightarrow S$, i.e. the pull-back of the relative tangent sheaf $\text{Der}_S A$ along the neutral section $S \hookrightarrow A$. The Hodge decomposition in the relative setting is

$$0 \rightarrow F^0 \rightarrow \mathcal{H} \otimes_{\mathbb{Z}_S} \mathcal{O}_S \rightarrow \text{Lie}_S A \rightarrow 0$$

with F^0 the 0-th piece of the Hodge filtration.

Note that when we fix $A \rightarrow S$ an abelian S -scheme, the equivalence above also implies the bijection between

- (1)' abelian S -subschemes of A ;
- (2)' sub-variation of rational Hodge structures of $\mathcal{H}_{\mathbb{Q}} = \mathcal{H} \otimes_{\mathbb{Z}_S} \mathbb{Q}_S$

which sends an abelian S -subscheme A' to $\mathcal{H}(A'/S)_{\mathbb{Q}}$. Conversely, given $\mathcal{H}'_{\mathbb{Q}}$ and object in (2)', $\mathcal{H}' := \mathcal{H}'_{\mathbb{Q}} \cap \mathcal{H}$ is a \mathbb{Z} -PVHS of type $\{(-1, 0), (0, -1)\}$ which defines an abelian S -scheme A' , and the evident map $\mathcal{H}' \hookrightarrow \mathcal{H}$ shows that A' is an abelian S -subscheme of A .

Deligne showed in [7] that a pure Shimura datum (\mathbf{G}, X) is universal in the following sense:

Theorem 4.2 (moduli of Hodge structures). *Let (\mathbf{G}, X) be a pure Shimura datum. Then the composition $w : \mathbb{G}_m \hookrightarrow \mathbb{S} \xrightarrow{x} \mathbf{G}_{\mathbb{R}}$ is a central cocharacter, independent of $x \in X$. For any (algebraic) representation $\rho : \mathbf{G} \rightarrow \mathbf{GL}_V$ over \mathbb{Q} such that $\rho \circ w : \mathbb{G}_m \rightarrow \mathbf{GL}_V$ is some central cocharacter $t \mapsto t^k \text{id}_V$ defined over \mathbb{Q} , the constant local system \mathcal{V} on X with fiber $V(\mathbb{Q})$ underlies a unique \mathbb{Q} -PVHS*

We also need the notion of (generic) Mumford-Tate groups, cf. [14].

Definition 4.3 (Mumford-Tate groups). (1) For $(V, h : \mathbb{S} \rightarrow \mathbf{GL}_{V, \mathbb{R}})$ a \mathbb{Q} -HS, its Mumford-Tate group is the smallest \mathbb{Q} -subgroup \mathbf{G} of \mathbf{GL}_V such that $h(\mathbb{S}) \subset \mathbf{G}_{\mathbb{R}}$. \mathbf{G} is connected. If the Hodge structure is polarizable, then \mathbf{G} is reductive. We write $\mathbf{G} = \text{MT}(h)$.

If W is a space of tensors on V , i.e. a subquotient of $\bigoplus_i V^{\otimes m_i} \otimes (V^{\vee})^{\otimes n_i}$ ($m_i, n_i \in \mathbb{N}$), then W is a \mathbb{Q} -HS for the natural action of \mathbb{S} if and only if it is stabilized by the natural action of \mathbf{G} by the tensor constructions. In particular, writing $W^{0,0}$ for the subspace of $W_{\mathbb{C}}$ fixed by $\mathbb{S}_{\mathbb{C}}$, then $W \cap W^{0,0}$ equals $W^{\mathbf{G}}$ the \mathbb{Q} -subspace fixed by \mathbf{G} , and this space is called the space of Hodge class of type $(0, 0)$ in W .

(2) Let S be a complex manifold, and $(\mathcal{V}, \mathcal{F})$ a \mathbb{Q} -VHS on S . We fix a model V for \mathcal{V} , i.e. a \mathbb{Q} -vector space such that for each $x \in S$ we have an isomorphism $V \cong \mathcal{V}_x$, and that for any $x, y \in S$, the induced isomorphism $\mathcal{V}_x \cong V \cong \mathcal{V}_y$ is induced by a prescribed path in S from x to y . Typically we fix a base point $s \in S$ and a path ℓ_x from s to x for each x , so that $V = \mathcal{V}_s$ and $V \cong \mathcal{V}_x$ is given by ℓ_x .

For each $x \in S$, we have the Mumford-Tate group at x , i.e. the Mumford-Tate group $\text{MT}(\mathcal{V}_x)$, which is identified as a \mathbb{Q} -subgroup of \mathbf{GL}_V via the isomorphism $V \cong \mathcal{V}_x$. There exists a countable union of analytic subspaces $\Sigma = \bigcup_n S_n$ and a \mathbb{Q} -subgroup $\mathbf{G} \subset \mathbf{GL}_V$ such that $\mathbf{G} = \text{MT}(\mathcal{V}_x)$ for any $x \notin \Sigma$. For $x \in \Sigma$, we have $\text{MT}(\mathcal{V}_x) \subsetneq \mathbf{G}$. \mathbf{G} is called the generic Mumford-Tate group of the \mathbb{Q} -VHS $(\mathcal{V}, \mathcal{F})$. When the \mathbb{Q} -VHS is polarizable, \mathbf{G} is reductive.

Remark 4.4. In general, the Mumford-Tate group of a \mathbb{Q} -HS (V, h) is a \mathbb{Q} -subgroup of $\mathbf{GL}_V \times \mathbb{G}_m$ so that the Hodge classes of (p, p) -type ($p \in \mathbb{Z}$) can be studied in the same way as in the above definition. In this paper we will only need Hodge classes of type $(0, 0)$ and the above definition suffices.

Example 4.5 (Kuga-Siegel case). Let $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ be a Kuga datum. Then the representation $\mathbf{G} \rightarrow \mathbf{GL}_V$ defines a \mathbb{Q} -VHS \mathcal{V} on X whose underlying local system is the constant sheaf of fiber $V(\mathbb{Q})$. By 4.2, this \mathbb{Q} -VHS is polarizable. Since the local system is constant, the polarization is given by some symplectic form $\psi : V \otimes V \rightarrow \mathbb{Q}(-1)$ which \mathbf{G} preserves up to similitude. Hence the Kuga datum (\mathbf{P}, Y) is equivalently given by a homomorphism of pure Shimura data $(\mathbf{G}, X) \rightarrow (\text{GSp}_V, \mathcal{H}_V)$.

The image of $(\mathbf{G}, X) \rightarrow (\text{GSp}_V, \mathcal{H}_V)$ is a subdatum $(\mathbf{G}', X') \subset (\text{GSp}_V, \mathcal{H}_V)$, which is also irreducible as (\mathbf{G}, X) already is. It follows immediately from the definition 4.3 that \mathbf{G}' is the generic Mumford-Tate group of \mathcal{V} on X .

Take a lattice $\Gamma_{\mathbf{V}}$ in $\mathbf{V}(\mathbb{Q})$, and a torsion-free congruence subgroup $\Gamma_{\mathbf{G}} \subset \mathbf{G}(\mathbb{Q})_+$ stabilizing $\Gamma_{\mathbf{V}}$, we get the connected Shimura variety $S = \Gamma_{\mathbf{G}} \backslash X^+$. The representation $\Gamma_{\mathbf{G}} \rightarrow \mathbf{GL}(\Gamma_{\mathbf{V}})$ defines a \mathbb{Z} -PVHS, as $\Gamma_{\mathbf{G}}$ acts on X^+ through the fundamental group of S , and the \mathbb{Q} -PVHS associated to it is obviously \mathcal{V} . The abelian S -scheme corresponding to this \mathbb{Z} -PVHS is exactly the fibred Kuga variety $M = \Gamma \backslash Y^+ \rightarrow S$ with $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$ and $Y^+ = \mathbf{V}(\mathbb{R}) \times X^+$.

In the rest of this section, we fix $\pi : M \rightarrow S$ an abelian scheme given by a fibred connected Kuga variety $M = \Gamma \backslash Y^+$, defined by the datum $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$, with a torsion-free congruence subgroup $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$, and $S = \Gamma_{\mathbf{G}} \backslash X^+$. If $M' \subset M$ is a special subvariety such that $\pi(M') = S$, then by 3.7 we see that M is a special subscheme of $M \rightarrow S$. We proceed to prove the inverse:

Theorem 4.6 (special subschemes vs. special subvarieties). *Let $M \rightarrow S$ be defined by $(\mathbf{P}, Y) = \mathbf{V} \rtimes (\mathbf{G}, X)$ and $\Gamma = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}$ as above. Let $M' \subset M$ be a special subscheme. Then M' is a special subvariety, and $\pi(M') = S$.*

Proof. The pure Shimura variety $S = \Gamma \backslash X^+$ is normal. For any non-zero integer $N \in \mathbb{N}$, write $\Gamma_{\mathbf{G}}(N) = \text{Ker}(\Gamma_{\mathbf{G}} \rightarrow \mathbf{GL}(\Gamma_{\mathbf{V}}) \rightarrow \mathbf{GL}(\Gamma_{\mathbf{V}}/N\Gamma_{\mathbf{V}}))$, then $\Gamma_{\mathbf{G}}(N)$ is a congruence subgroup in $\Gamma_{\mathbf{G}}$, and the base change

$$\pi_N : M_N = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}(N) \backslash Y^+ \rightarrow S_N = \Gamma_{\mathbf{G}}(N) \backslash X^+$$

is an abelian S_N -scheme in which the N -torsion subgroup split, i.e. $M_N[N] = \coprod_v M_N(v)$, where

- the disjoint union is indexed by $\frac{1}{N}\Gamma_{\mathbf{V}}/\Gamma_{\mathbf{V}}$, which is the N -torsion subgroup of $\Gamma_{\mathbf{V}} \backslash \mathbf{V}(\mathbb{R})$;
- for $v \in \mathbf{V}(\mathbb{Q})$, $M_N(v)$ stands for the special subvariety defined by $(v\mathbf{G}v^{-1}, vX; vX^+)$.

Note that for general $v \in \mathbf{V}(\mathbb{Q})$, the special subvariety $M_N(v)$ only depends on the class of v in $\Gamma_{\mathbf{V}} \backslash \mathbf{V}(\mathbb{Q})$, and the resulting special subvariety is $\Gamma' \backslash vX^+$, with $\Gamma' = \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}} \cap v\mathbf{G}(\mathbb{Q})_+v^{-1}$, which is equal to $v\Gamma_{\mathbf{G}}(v)v^{-1}$ with

$$\Gamma_{\mathbf{G}}(v) = \{g \in \Gamma_{\mathbf{G}} : g(v) - v \in \Gamma_{\mathbf{V}}\}.$$

Since $\Gamma_{\mathbf{G}}(N)$ is the kernel of $\Gamma_{\mathbf{G}} \rightarrow \mathbf{GL}(\Gamma_{\mathbf{V}}/N) \cong \mathbf{GL}(\frac{1}{N}\Gamma_{\mathbf{V}}/\Gamma_{\mathbf{V}})$, we see that $g \in \Gamma_{\mathbf{G}}$ always fixes the class of v modulo $\Gamma_{\mathbf{V}}$ when $v \in \frac{1}{N}\Gamma_{\mathbf{V}}$, hence $M_N(v) \cong \Gamma_{\mathbf{G}}(N) \backslash X^+ = S_N$. This isomorphism is actually the Hecke translation given by v , using $v(\Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}(N))v^{-1} \cong \Gamma_{\mathbf{V}} \rtimes \Gamma_{\mathbf{G}}(N)$, cf. 3.6 and 3.5.

By the definition of special subschemes, it remains to show that every abelian S -subscheme of $M \rightarrow S$ is a special subvariety M' such that $\pi(M') = S$. It suffices to treat the problem for a sufficiently small level $\Gamma_{\mathbf{G}}$, so by shrinking $\Gamma_{\mathbf{G}}$ we may assume that the sheaf of endomorphism algebra $\mathbf{End}_S(M)$ is constant, as we have seen that over the integral normal scheme S the sheaf is locally constant and its generic fiber is a finite rank \mathbb{Z} -algebra. Since we only need to study the neutral component of the kernel of endomorphisms, we may replace $\mathbf{End}_S(M)$ by the isogeny algebra $\mathbf{End}_S^{\circ}(M)$, which is also constant.

Passing to isogeny from the equivalence in 4.1, the sheaf $\mathbf{End}_S^{\circ}(M)$ is the same as the endomorphism sheaf of the \mathbb{Q} -PVHS $\mathcal{H}_{\mathbb{Q}} = \mathcal{H}(M/S) \otimes_{\mathbb{Z}_S} \mathbb{Q}_S$, namely the sheaf associated to the Hodge classes of type $(0, 0)$ in $\mathbf{End}(\mathcal{H}_{\mathbb{Q}}) \cong \mathcal{H}_{\mathbb{Q}} \otimes_{\mathbb{Q}_S} \mathcal{H}_{\mathbb{Q}}^{\vee}$ with $\mathcal{H}_{\mathbb{Q}}^{\vee}$ the \mathbb{Q} -PVHS dual to $\mathcal{H}_{\mathbb{Q}}$. Since $\mathcal{H}_{\mathbb{Q}}$ is given by the representation $\mathbf{G} \rightarrow \mathbf{GL}_{\mathbf{V}}$, $\mathbf{End}(\mathcal{H}_{\mathbb{Q}})$ is given by the tensor representation $\mathbf{G} \rightarrow \mathbf{GL}_{\mathbf{End}(\mathbf{V})}$, and the $(0, 0)$ part corresponds to the trivial subrepresentation $\mathbf{End}(\mathbf{V})^{\mathbf{G}}$. So the constant sheaf $\mathbf{End}_S^{\circ}(M)$ is associated to the vector space $\mathbf{End}(\mathbf{V})^{\mathbf{G}}$.

Let M' be an abelian S -subscheme, realized as the neutral component of some $\phi \in \mathbf{End}_S^{\circ}(M)$. We thus identify ϕ as an element of $\mathbf{End}(\mathbf{V})^{\mathbf{G}}$, and it follows from the equivalences 4.1 and the characterization of abelian subschemes via sub- \mathbb{Q} -PVHS that M' corresponds to the \mathbb{Q} -PVHS \mathcal{H}' given by \mathbf{V}' which is the kernel of $\phi : \mathbf{V} \rightarrow \mathbf{V}$. Clearly \mathbf{V}' is a subrepresentation of \mathbf{G} in \mathbf{V} , and for any $x \in X$, the action of \mathbb{S} on \mathbf{V}' through x makes \mathbf{V} a sub- \mathbb{Q} -HS of \mathbf{V} , hence is of type $\{(-1, 0), (0, -1)\}$. We obtain a Kuga subdatum $(\mathbf{P}', Y') = \mathbf{V}' \rtimes (\mathbf{G}, X)$, which defines an abelian S -subscheme, whose associated \mathbb{Q} -PVHS is the one given by the action of \mathbf{G} on \mathbf{V}' . Therefore this abelian S -subscheme is equal to M' , and M' is a special subvariety, faithfully flat over S under π . \square

We immediately get the desired variant of the Manin-Mumford conjecture in the Kuga setting

Corollary 4.7. *Let $\pi : M \rightarrow S$ be a Kuga variety fibred over a pure Shimura variety S as an abelian S -scheme. Let (M_n) be a sequence of special subvarieties faithfully flat over S , i.e. $\pi(M_n) = S$ for all n . Then the Zariski closure of $\bigcup_n M_n$ is a finite union of special subvarieties faithfully flat over S .*

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